THE MATHEMATICS STUDENT

A Quarterly Dedicated to the Service of
Students and Teachers of Mathematics in India

OFFICE OF THE REGISTRAR OF BOOKS
27 APR 1935
MADRAS

Edited by

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Published by

The Indian Mathematical Society

in March, June, September and December

Vol. II. 1934.

Printed by

SRINIVASA VARADACHARI & CO.
190, MOUNT ROAD, MADRAS.

Annual Subscription : Rs. 6.

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OFFICE OF THE ... 15 MAR 19..

THE MATHEMATICS STUDENT

Volume III] SEPTEMBER 1935 [Number 3

THE SPACE CONCEPT—(*Contd.*)

BY

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II. Metrical

1. In the previous article we set before ourselves the question: What are those simple concepts into which all the concepts that we have of spatial figures can be analysed, and with which we can build up again the complex concepts of a spatial figure? We have indicated the elementary concepts with which the Science of Descriptive Geometry can be built up; but this is not the whole science of spatial figures. Let us therefore enquire into the axioms necessary for the rest of Geometry, which, for distinction, goes under the name of Metrical Geometry.

2. Firstly, it must be noted that the axioms we have already mentioned are pre-supposed in any further development of Geometry. The question now is: what are the additional properties of our spatial figures which we must assume in order to render possible the further development of Geometry? We bring in the concept of quantity, and this addition widens the subject, makes it more complicated, and brings it nearer to the concrete. Like all knowledge that is to be of any practical use, Geometry should be abstracted from the concrete, but in such a way as to allow us to retrace our steps, as it were to the concrete; in other words our knowledge should be re-applicable to Nature.

3. From our sense-perceptions we abstracted the concepts of spatial elements, and viewed each under three aspects; we obtained the ideas of order and of one-to-one correspondence. We have shown how spatial figures can be reduced to the spatial elements. Now to the ideas already brought forward we add that of quantity. This itself is abstracted from our sense-perceptions of material things by prescinding from all causal properties, active and passive, and hence also by prescinding from the particular physical or chemical nature of substances. Our pre-suppositions of Descriptive Geometry are necessary. We take quantity to mean the property of having integral parts, i.e. parts of the same type as the whole and making up the whole, of which parts one is ...er and each one is exclusive of every o... For our purposes, this ...ty is sufficient. The spatial ... be affected by ...lered to have parts wh... ...elf be considere...

3·1. Our spatial figures we have already reduced to spatial elements, and if the concept of quantity can be applied to these latter, then, consequently and in so much, spatial figures may also be considered as affected by quantity, or as "metrical". If quantity were to be applied to spatial figures without the use of the elements of Descriptive Geometry, then we should need other qualitative elements into which the metrical figures can be resolved, because in the idea of quantity is pre-supposed a common trait in the thing and its parts, and the possibility of identifying as well as of differentiating the parts. The basis for the latter we have already chosen; and, retaining the same, it remains to be shown how the concept of quantity is associated with our spatial elements.

3·2. Quantity itself is a pre-requisite for the consideration of the causal properties of things; and so the whole of Geometry, Descriptive as well as Metrical, is a pre-requisite for the possibility of the Physical sciences. Geometry itself is not a physical science.

3·3. The concept of quantity made use of in Geometry is not the whole complex concept; but only a part of it, and very much more abstract. Therefore it will be advisable in what follows to speak of spatial "magnitudes" rather than of spatial quantities. Quantity is not directly applicable to Time, as we shall see later, but only through physical concepts or geometrical magnitudes. When associated with Mass, causal as well as temporal effects must be taken into consideration. Moreover, as we prescind from matter and any particular kind of "substance", the question of divisibility will not arise in the same way as it does with matter. In the laboratory sciences continued sub-division will eventually lead to a full-stop: the elemental crystal, or the molecule, or the atom, or the electron; so that we cannot admit the possibility of indefinite sub-division. We find no such restrictions in Geometry as we have in Physics or Chemistry, and we are at liberty to admit indefinite subdivision, treating infinity as an exceptional kind of mathematical number, in the same way as Zero should also be treated; that is, with certain reservations as to its subjection to the rules of number in general. A property of number is that it is greater than some one number and less than some one number. The meaning of "greater than" and "less than" can be taken not only as "directions", but also, after we have given an explicit meaning to numbers, in the quantitative sense. Zero can be considered to belong to the system of positive numbers in as much as it has this property partially, i.e. it is less than every other positive number; the zero space-element can be considered as a part but itself not having parts. Infinity can be said to belong to the system of positive numbers in as much as it is greater than every other number. The "Complete" space-element can be considered as having parts but itself not a part. The zero spatial element is a magnitude in only a partial sense, and so is the complete space-element. These ar[illegible] necessary restrictions in our concepts [illegible] magnitude. The [illegible]antity mentioned above leads [illegible]

and of putting the resu[illegible]

4. The General Axioms of magnitude (Euclid's Axioms 1—7 and 9) only express abstractions made from our experience in dealing with concrete things the same abstractions that are made helping us to build up the concept of the number-system and the Laws of Arithmetic. They give an explicit metrical significance to the expressions "greater than" and "less than." Euclid's eighth axiom defines "equality": "magnitudes which can be made to coincide with one another are equal." Without some definition for equality we should have no means of comparing or measuring a magnitude, and putting the result into correspondence with the Number-System.

5·0. Quantity is not associated in the first instance with the elemental point, except in the manner indicated below. Quantity is associated with the elemental straight line, and is called linear magnitude or "length."

5·1. We have already seen the straight line in its threefold aspect. It is in the aspect of a spatial element that we attribute quantity to it. It is not attached to the straight line considered as a mere descriptive property. However, the other aspects of the straight line cannot be detached from it.

5·2 We considered the straight line also as the aggregate of points satisfying a common descriptive property. If A and B are two distinct elemental points then the base l through A and B is a descriptive property common to A and B, and to other elemental points in common with A and B. By means of our concept of ORDER we can recognise an elemental point as lying between A and B or not. That elemental straight line, the descriptive property of which applies to A and B and to every point C lying between A and B and does not extend further, is called the *Segment* AB of the base l. We can consider the straight line as subject to spatial magnitude for it can have parts or segments of the same type as the whole, i e. with the same common property, the base l; which parts can be distinguished from each other and from the whole, as already indicated in In. 1·11.

5·3 Indefinite subdivision will lead to zero magnitude in the sense above indicated, with the definite concept of the zero straight line as a segment terminated by two descriptive points *with no elemental point between* them. A non-zero segment has parts, but a zero segment can only be a part and have none itself. If the descriptive terminal points be considered also as elemental, we can enumerate the points in a zero straight line as two only; but not in a non-zero segment. The descriptive base l itself may be considered as elemental not defined by descriptive terminal points; it is our complete linear magnitude, in the sense that we can consider segments of it as its parts, but itself as not being a part, since it is the totality itself. For if A and B be any two points on it we can always find a point C having the same descriptive property l in common with A and B, and not lying between [illegible] B on l; so that AB will be a part of AC or of CB, but not vice-vers[illegible] points, e. g. A, B, C, serve as *descriptive* properties af[illegible]

the parts as well as for their differentiation. In the segments AC and CB the point C common to both is also descriptive; it is not a spatial element. The points of a segment are in the first instance descriptive points, but they may be taken also as elemental points; in this case the segment defined by them may be considered as a spatial figure, or a figure resolvable into more than one spatial element. We shall, however, avoid using the expression "linear figure" to mean only a segment or line joining two elemental points, as ordinarily it has a different significance.

5·4 The zero of linear magnitude can be called the "elemental segment" or "zero-segment." We can identify elemental segments and distinguish them by means of their descriptive points. We can also identify them continuously with the additional aid of a descriptive point common to any segment and the next taken in order. This continuous identification will be called a "motion," and we shall speak of "moving along a straight line" in this sense.

6·0 Motion in Geometry can only be an intellectual operation. We do not depend on sense-experience or physical operations. When we say "move from point A to point B," or "move A to B," or "transpose A to B," we mean that we identify A and B by recognising a common property between points as we mentally proceed along them from A to B, in the descriptive straight line on which A and B may lie, or in the series of elementary zero-segments to which the contour joining A and B can be reduced.

6·1 The descriptive properties of the point O on the straight line l can be made use of to distinguish the points of l into two classes. If on setting out to move from O along the straight line to P, we pass through Q, or on going from O to Q we pass through P, then P and Q can be considered as belonging to the same class. When neither the one nor the other is the case, they can be considered as belonging to different classes.

6·2 Again two descriptive points A and B will distinguish the points of the straight line l by separating the points between A and B from those outside; and the segment AB will also distinguish the points outside it into two classes in this manner: if from any point P we go to A, then P belongs to one class or the other according as we traverse B in this process or not.

6·3 Thus we see that we have the possibility of divisibility in the straight line element, giving us the possibility of associating quantity with it. We have already associated the zero-concept of the absolute number system with the straight line; but to put the other numbers into a definite correspondence with the metrical linear element we must consider quantity as subject to definite comparison, in a manner corresponding to the comparisons instituted between the numbers. The essential property of the integral numbers is the invariant relation between any integer and the next in order, the arithmetical difference being unity. This is the b[illegible] the introduction of the notion of the rationals and through [illegible]'s, as well as for the arithmetical laws. We [illegible] least an invariant relation between the

partial segments into which a whole is divided. In other words we must define when two segments are said to be "equal," that is, identifiable in some definite metrical way.

6·4 The segment AB is said to be equal to the segment CD on the same base l when one can be exactly superposed upon the other. We assume now that any segment has an *inherent descriptive* property, such that if the segment be given this metrical property all the internal and external relations of the segment are fixed; and when this property or magnitude alone is given only the internal relations of points belonging to the segment are fixed.

7·0 **The Axiom of Length.**—An inherent property of a figure is one so connected with it that it is defined and determined only by the figure itself with no reference to extraneous elements. We assume that each segment has an inherent metrical property called its LENGTH which is the integral sum of the quantitative parts making up the whole segment. It is intimately connected with the descriptive terminal points of the segment. If A and B are fixed all the extrinsic and intrinsic properties of the segment are fixed; but if only the intrinsic properties be given, its position is not fixed, for the relation of its points to extrinsic points are not known. If, however, one terminus A be given as well as the intrinsic property, then the other terminus is also determined, and consequently everything connected with the segment is known. This is the characteristic of the intrinsic property "length." Two segments may have the same length but still otherwise be distinct. *When we equate two segments we are only identifying their intrinsic properties.* If A is given and the length l' of the segment AB, then the segment is fixed with all its intrinsic and extrinsic properties; so also if C be given and the length l'' of the segment CD. When, therefore, A is moved to C, using the Axiom of homogeneity, (see n. 6.0) then if B coincide with D, the segment AB and CD are said to be "equal" in magnitude $l' = l''$. This *movement of A along with the intrinsic* l' into the position of C, so that AB falls along CD is called "Superposition." Without this Axiom of Length we have no means of comparing linear magnitudes other than recognising the mere greater than or less than, which refer rather to order than to magnitude. We wish to know also the "how many times", i.e. to be able to measure. If one segment can be superposed upon another, they are equal in magnitude, and conversely, if two segments are equal in magnitude one can be superposed upon the other. (Note that this is a "definition" of equality, and not a "test." Whether the intrinsic property be altered by physical factors such as change of place, velocity or time, does not concern us at present; for our "motion" is purely *intellectual*, prescinding from all causal properties; and our definition of equality is left to our choice.)

8·0 It is to be noted that order is maintained between the descriptive elements of the two segments; A corresponding to C and B to D, the descriptive base l being common to both, and points between A and B corresponding to points between C and D. If on moving AB along the base we can identify A and D, B and C, then the segment AB is said to be equal to the segment DC.

This enables us to place our spatial linear magnitude into a closer relationship with the number-system. Between the numbers 4 and 5 and the numbers 4 and 3 there is unit interval, but the difference in the nature of this interval necessitated the introduction of the positive and negative sign as descriptive symbols. Here also in spatial linear magnitude the segment AB will be said to be equal to the segment CD in the first case considered, and to the segment DC in the second case; to the negative of DC in the first and to the negative of CD in the second. In the first case we cannot make A correspond with D; in such a way that B will correspond with C; but we can make A to correspond with C in such a way that B will correspond with D, and points between A and B with points between C and D. We therefore write AB = CD, and AB = − DC. Thus it will also be seen that AB = − BA. If it were immaterial with which of the points C or D we may identify A or B, then the segments are just said to be equal in undirected magnitude.

9·0 **The Axiom of Free Mobility.**—In the first case to make A correspond with D and B with C we should have to move the segment AB out of the base *l*, using the descriptive plane for the motion. This is possible; for it will be noted that the "motion" of superposition or identification implies that the inherent descriptive property or length of the segment AB is carried over by the common descriptive property of the base on which both AB and CD lie, and so the lengths of both may be identified, that is "equated." If, therefore, the two segments AB and CD have not a common descriptive property, the straight line *l*, they might still have another common descriptive property, the plane *p* on which both lie. In this case also we may superpose and obtain our condition for "equality"; but we cannot bring magnitudes into correspondence with the Arithmetic Continuum as composed of positive and negative numbers, unless the segments have the same base in common. It may be that the two segments have not only no straight line base in common but also no descriptive plane in common. In this case we transpose AB in the plane common to AB and C to make the point A coincide with C, and then transpose AB in the plane common to CD and B in its new position, whatever it be; so that B falls on D. Thus the plane and the straight line give us the bases for the identification of the descriptive point A with the descriptive point C, such that the descriptive point B will be correspondingly identified with the descriptive point D; in short, they are the bases for equating two linear segments.

9·1 This sort of identification is also expressed by the words "superpose" and "superposition." When the segments belong to the same base the concepts of positive and negative magnitude can be introduced, as we have seen, but not otherwise.

10·0 Having given a meaning for the equality of linear magnitudes, it is easy to introduce the ideas of measurement, of ratio, of a unit of length, of the arbitrariness of such a unit or rather of the fixing of it at our convenience, of commensurable magnitudes, and by extension by the aid of the notion of limits also of incommensurable magnitudes. All this sets up a closer

correspondence between linear magnitudes and the Arithmetical Continuum. It is sufficient that we have indicated the basis for these developments. Let us merely observe that "length" in the sense of measured linear magnitude, that is, given as so many units, is merely a comparison.

11·0 With regard to curves, they are reduced to spatial elements, and the integral sum of the elemental straight line segments so obtained is called the length of the curve.

11·1 Again any spatial figure is reducible to spatial elements, and two figures, X and Y, are said to be "linearly equal" when *all* the linear magnitudes intrinsically connected with X are equal each to each to the corresponding linear magnitudes intrinsically connected with Y.

12·0 **Rotation.**—If OP be a segment, of which O is given, also the inherent property of its length and the plane on which it lies, then P can be on any one of the straight lines that makes up the point O. (See I n 1·14) Moreover these points themselves make up a figure in which we have continuity. (see I n 5.3) It is a closed figure (see 13.0 below) called a "circle" and is also descriptive of the elemental points in the plane, distinguishing them into two classes, not only as lying within or outside the circle, but also such that their distances from O are less than or greater than the given length. The length of the closed contour from any point A back to itself can be calculated (see 11.0). If P move round the contour (see 6.0), then we say that P rotates about O at a constant distance, called the radius, in the plane p. Consider now the straight line l bounded at one end by O. If any point P on it rotate about O in the plane p then any other point X on l also rotates about O, since the lengths OP and OX remain the same for all positions of l; the corresponding positions of P and X will in all positions be collinear with O. We may therefore say that the whole straight line l rotates about O in the same plane p. P describes a curve $P\,P_1$ from its initial position P, and X a curve $X\,X_1$ from its initial position X; the greater $P\,P_1$ be in length, the greater will $X\,X_1$ also be. Moreover they can be shown to be proportional, and each proportional to the radius. Hence we can associate magnitude with the "rotation" of a straight line about a point in it: it varies directly as the length of the arc traced out by any one point on the straight line and inversely as the radius vector to that point.

12·1 **Angle.**—An angle can strictly be associated only with straight lines. The notion of an angle between curves, or planes or surfaces are all reducible to that of the angle between linear elements. The notion of an angular magnitude can be analysed into a comparison of linear magnitudes by the process indicated above. Given two elemental straight lines with a common descriptive point, and consequently also a common descriptive plane, we associate a magnitude with the figures so formed by considering the amount of rotation required to move from one straight line to the other about the common point. This is rendered possible by our ability to measure a curve, to have a ratio of two lengths, to equate ratios and therefore also to measure them.

These notions simplify our analysis of spatial figures by saving us the trouble of a complete analysis into its descriptive and metrical elements. The condition for the equality in "linear magnitude" (see 11.1) of two plane figures is thus simplified. Superposition of one angular magnitude AOB upon another PXQ is now possible, signifying the identification of points of one straight line with those of another; *e. g.* O with X, points on OA with those on XP, and those on OB with those on XQ, which is possible since the same amount of rotation will take a point C on OA or XP into the position of OB or XQ. It is an easy process to associate sign also with angular magnitudes in the same plane.

12·2 In connection with angular magnitudes an assumption may be made that any two complete rotations are equal. This would follow, however, from the definition of equality by superposition. Defining a right angle as a quarter of a rotation and applying Euclid's Axiom VII twice we should get "All right angles are equal" (Euc. Ax. XI).

13·0 **Area.**—The question now arises whether we can associate the notion of magnitude with that of the elemental plane in a manner analogous to its association with the straight line. A "*plane figure*" is one whose descriptive elements are straight lines, or curves reducible to straight lines, all of which considered as spatial elements have a descriptive plane. Such a figure can be considered as descriptive of and defining a portion of an elemental plane and serves as the basis for distinguishing elemental points in one part of the plane from those in the rest of it. If we move along the figure continuously (see 6.0) and arrive at the starting-point without having to retrace our steps wholly from any other point to the starting point, the figure is a "closed" one, and the boundary is called its "contour". A closed contour will distinguish the points of a definite portion of the plane from the other points of the plane. To move in the plane from a point A within the figure to a point B outside it we must cross the contour; this is not so if both the points were within the figure or both outside. We cannot "identify" (seen 6.0) a point lying in the portion A enclosed by the contour C with a point B situated elsewhere, for the distinctive contour C would check our progress: in other words, no straight line, or curve reducible to straight lines can connect up a point A within C and the point B outside it without meeting the differentiating descriptive contour C, which prevents any such identification: unless, of course, we move out of the plane. Thus the elemental plane can be considered as having parts, each part of the same kind as the whole all having a common property, the descriptive plane, and distinguishable from each other by means of descriptive closed curves.

13·1 Superposition of plane figures can be used to define what is meant by the equality of two plane figures. Moreover by dividing the portions of a plane defined by two closed contours into elementary superposable figures the magnitudes of the figures can be compared. Magnitude associated with the plane is called "area" or "areal" magnitude. It is the integral sum of the parts making up the plane bounded by the figure. It is an intrinsic property of the figure associated with the plane on which the figure lies and intimately

connected with the linear and angular magnitudes of the figure. The "limiting Zero" of area is that of a closed figure, and is not a point or a straight line; the figure is considered in the first instance as the descriptive property of a portion of the plane. So also the "Complete plane" can be considered as having parts but itself not a part.

13·2 If two figures in the same plane are equal with regard to linear magnitude in all the corresponding linear elements, then either one can be superposed upon the other exactly with or without motion out of the plane. This observation enables us to associate the positive and the negative sign with a real magnitude. A convention is easily introduced to distinguish the sign associated with the area. If the figures be not in the same plane, a further convention is employed. The plane itself can be taken as descriptive of points in space, and distinguishing them into two classes, points on one side of it, and points on the other. If points on one side of the plane be considered as positive points, and those of the other negative, that is, the plane be regarded as having a positive side and a negative side, then according to the usual convention for the sign of an area, the same area will be positive on one side, and negative on the other. Two areas, one on one plane and one on another can also thus be compared with regard to sign.

14·0 **Volume**—We can associate quantity also with solid figures of space, which we call "volumetric magnitude" or "volume". A solid figure is one the descriptive elements of which are planes, or are reducible to planes, which planes, considered as elements, have only the common property of belonging to space. (See I n 2.1) We have already spoken of the homogeneity that pervades in general all spatial elements. Without this we have no means of identifying or equating volumes, or of measuring them; in fact volume can have no meaning, and we could not associate magnitudes with "space".

14·1 A "solid figure" is one whose descriptive elements are planes, or surfaces reducible to planes, all of which considered as spatial elements have only the homogeneity of space in common. *A closed solid* figure is one, the descriptive planes of which would so distinguish the elemental points of a part of space that we could not identify (see n 6.0) any one point A within it with any point B outside it without coming up against the distinctive barrier composed of the descriptive planes making up what is called the "solid surface". In the case of the closed plane figure a means of identification of the point A within it and the point B outside it was still left; namely by moving out of the plane so as to avoid the barrier formed by the contour. We have no corresponding possibility here. However, since we are able to superpose one solid figure upon another we have the possibility of defining the equality of volumes, and this possibility provides the basis for the association of magnitude with spatial figures. The "volume" of a closed solid figure is the integral sum of the parts making up the space bounded by the figure. It is an intrinsic property of the figure associated with space and intimately connected with the linear, angular and areal magnitudes of the figure. In

order to associate this magnitude with the whole Continuum of positive and negative numbers a convention is adopted, which of course is not obtained in a manner analogous to that by which the convention used in the case of linear figures and plane figures was obtained. When two spatial figures are equal in all linear magnitudes correspondingly associated with them, they may or may not be superposable, and in the latter case, we cannot get out of space to make them so; however, they are said to be equal in magnitude but opposite in sign. The convention adopted depends only on the way the figure is named. (See any text-book on Vectors.) This naming is usually done along the elemental straight lines or curves common to the descriptive surface, two by two.

14·2 The "limiting zero" of volumetric magnitude is that of a closed solid figure with bounding descriptive planes. It is quantity in a partial sense; it is a part but itself has no parts. The number of zero-volumes in a non-zero solid figure is mathematically infinite, but the limiting sum is the volume of the non-zero solid figure. "Complete" space is also quantity in a partial sense: it has parts but itself is not a part. We have seen that the limiting-zero of mathematical magnitude does not correspond to any physical zero of quantity except non-entity, so also the mathematical concept of the whole of space does not correspond to any physical concept, except the whole universe of observation.

15·0 The two descriptive points of the zero-segment can be considered also as elemental points determining the one descriptive straight line, with the precautionary observation that there is no elemental point between them: for if there were one then the segment would have parts and not be the zero-segment. Similarly the descriptive straight lines (or closed curve reducible to straight lines) of the zero-area can be considered as elemental straight lines only with the precaution that there is no elemental point within the closed figure; for otherwise it would have parts. So also the descriptive planes (or closed surfaces reducible to planes) of the zero-volume can be considered as elemental planes in themselves, with the precaution that there is no elemental point within the closed solid figure. The zero-magnitude of length, of area or of volume may be considered as a point only in the sense that the elemental point will locate it. The descriptive properties, however, attached to each zero of magnitude cannot be discarded.

16·0 We shall now give the concepts of linear, areal, and volumetric magnitude in summary form.

(1) The limiting zero of linear magnitude is the segment bounded by descriptive points with no elemental point within the segment.

The limiting zero of areal magnitude is the area bounded by descriptive straight lines with no elemental point within the closed plane figure.

The limiting zero of volumetric magnitude is the volume bounded by descriptive planes with no elemental point within the closed solid figure.

(2) The least number of descriptive points required to define a segment is two.

The least number of descriptive straight lines required to define an area is three.

The least number of descriptive planes required to define a volume is four.

(3) Two descriptive points determine a straight-line segment with all its extrinsic properties and its intrinsic property of length.

Three descriptive straight lines, not more than two at a time belonging in common to one elemental point, determine the extrinsic properties of a plane figure as well as its intrinsic property of area.

Four descriptive planes, of which not more than two belong to a common straight line and not more than three belong to a common elemental point, determine a volume.

(4) The basis for the introduction of magnitude in connection with the straight line is the homogeneity of the straight line; in connection with the plane it is the homogeneity of the plane; in connection with spatial figure it is the homogeneity of space.

(5) Directed magnitude, i. e. magnitude put into correspondence with the Arithmetical Continuum of positive and negative numbers, is possible with the straight line. It is possible also with areas, but the direction is governed by motion along a contour. It is possible also with volumes, the direction being governed also by motion along a contour.

(6) A straight line will divide the points of a plane such that a segment will subtend a positive angle at any point on one side of the straight line and a negative angle at any point on the other side of it. A plane will divide the points of space in such a way that an area will appear positive with respect to points on one side of the plane and negative with respect to points on the other side of it, according to the convention. But with regard to space itself we cannot proceed further.

(7) Two segments of the same length are superposable. If the magnitudes are equal but opposite in sign the motion of superposition will take us out of the common straight line on which the segments lie, but remaining in the same plane.

Two areas of closed curves whose intrinsic linear properties are equal, are superposable in the same plane when the magnitudes of the areas are the same with the same sign; otherwise the superposition will carry us out of the plane.

Two volumes of closed spatial figures, whose intrinsic linear properties are equal, are not always superposable, for we cannot go out of space. If not superposable the volumes can be considered to have opposite signs. A right-handed and a left-handed screw are not superposable, although all the linear measurements of the one are equal to the corresponding measurements of the other.

(8) Angular magnitude is reducible to the ratio between two linear magnitudes. The notion helps us to superpose figures.

(9) The basis for the equation of areas and of volumes are linear magnitudes.

17·0. In so far as correspondence has been established between spatial magnitude and the number-system analytical methods are perfectly legitimate, involving notions of the limiting zero and infinite magnitude and of limit-summation, with due precautions.

18·0. **Continuity**.—The notion of continuity that we obtained in Descriptive Geometry now takes on a quantitative aspect. The elemental zero-segment has length in so far as the zero-segments of a straight line all put together make up the non-zero segment (See n 5.3) although when compared with the whole each zero-segment is negligible mathematically, and may be treated as absolute zero. Thus the concept of infinitesimals of mathematical analysis can be adopted with regard to spatial magnitude; because in fact measured quantity has been put into close correspondence with the whole number-system. The straight line l is said to be continuous in the sense that, given any linear magnitude e and any point A, we can find at least one point B on the straight line such that the length AB is less than e. The same idea is extended to curves, to areal figures, to surfaces and volumes.

19·0. **Dimensions**.—The "number of dimensions" of a descriptive property is the least number of measurements necessary in order to determine uniquely the position of a point having that descriptive property.

19·1. We have seen that in order to determine:

a segment, we must know at least TWO distinct points A and B in the straight line base.

an area, we must know at least THREE distinct points A, B, C, not all in the same straight line.

a volume, we must know at least FOUR distinct points A, B, C, D, not all in the same plane, and therefore also no three in the same straight line.

19·2. In order to fix the position of a point P we must know: a straight line l descriptive of it, one point O on it and one length OP. Hence the least number of the other points required to fix a point P on a straight line is ONE, O, and the least number of measurements necessary is also ONE, OP. Therefore the straight line is said to be of ONE dimension.

19·3. In order to fix the descriptive straight line l we must consider it as elemental, and we need to know then a plane p descriptive of it, and two descriptive points of it. Hence the least number of other points required to fix a point P in a plane p is TWO, A and B, and the least number of measurements necessary is also TWO, AP and BP. Hence a plane is said to be of two dimensions.

19·4. In order to fix the descriptive plane p we must know two straight lines having a point in common, or a straight line and a point outside it, or

three non-collinear points. Hence the least number of other points required to fix a point P in space is THREE A, B, C, and the least number of measurements necessary is also THREE, AP, BP, CP. Hence we say that space is of three dimensions.

19 5. It is not to be supposed that space taken in any sense whatever is of three dimensions. We have only shown how the preliminary postulate of three spatial elements (See I n 5.4) after the introduction of quantity led to this conclusion. Three-dimensional space is the only geometrical space we have considered so far. By extension it is possible to conceive four descriptive elements not reducible to each other but inter-related, and consequently deduce a four-dimensional "continuum" we use a word other than "space" for it; and we can build up by analogy even higher-dimensional continua. When considering "events" we need a fourth element, "time"; that is, we place events in a continuum of four dimensions. But just as the elements themselves that we have to deal with are not dimensions, so time is not a dimension but an element necessitating an additional measurement in order to fix an event.

20·0. **Degrees of Freedom.**—A point restricted to a straight line is said to have ONE degree of freedom. If a linear magnitude be given fixing a point in the straight line, it is said to have no "freedom". If the magnitude be not directed there will be two points satisfying the condition, but they are determinate points.

20·1. A point restricted to a plane is said to have TWO degrees of freedom. If two linear magnitudes be given fixing it then it has no freedom; if only one linear magnitude be given, it has one degree of freedom, and it can have a descriptive property called a "locus", which is the descriptive curve of the point satisfying the condition.

20·2. A point in space is said to have THREE degrees of freedom. If three linear magnitudes be given fixing it, it has no freedom; if two only it has one degree of freedom, and then we have a locus as before, but which is not necessarily a plane curve; if one linear magnitude only be given, the point has two degrees of freedom, and there is a descriptive surface, also called a locus sometimes, satisfying the condition.

20·3. Figures whose elemental points are devoid of freedom are said to be "rigid".

21·0. Area is magnitude associated with the plane, and volume is magnitude associated with space. These magnitudes can, however, be brought into relation with linear magnitudes, because they were defined with the help of linear magnitude. We have seen that for a closed plane figure the least number of descriptive straight lines is three. This would give us three elemental points and three segments, and three measurements would be required to fix its area. These are transformable into two linear magnitudes and one angular magnitude. Hence, if the standard triangle be considered as right-angled we need admit only two linear measurements to determine the area of

an elemental triangle. Instead of the right-angled triangle the elemental figure of area may be taken as the rectangle. We need not go further into this point. Suffice it to say that we may consider the area of a plane figure as the "product" of two numbers put into correspondence with two lengths, that is, an area may be measured as the "product of two lengths". The resulting number can be taken as proportional to a length, and with a suitable convention an area can be denoted by a vector. (See any Text-Book on vectors.)

21·1. Similarly the least number of points fixing a solid figure is four, and the least number of segments involved is six. If these six are given the figure is "rigid". They can be reduced to three linear and three angular magnitudes. By fixing the angular magnitudes of the figure circumscribing the elemental volume, we have a figure of a standard type involving three linear magnitudes. Hence a solid figure has a volume proportional to three linear magnitudes, and can be considered as the "product of three segments". This point is developed in any work of Geometry.

21·2. We have put spatial magnitudes into correspondence with the system of numbers including zero. Just as we have different orders of infinitesimals so we have different orders of infinities. "Infinity" itself is not a number just as "infinitesimal" is not a number; but a significance is given to the phrase "tends to infinity" and to operations to infinity. Infinite and Infinitesimal are reciprocal adjectives: infinity and zero may be taken as their abstract forms. If A is an infinitesimal compared to B then B is infinite compared to A. A linear non-zero segment we call a finite straight line. It is infinite compared to a zero-segment. Let us examine further the notions of the complete space-element, linear planar or spatial.

21·3. The complete straight line which has parts and itself is not a part, when subjected to measurement, i. e. compared to a finite segment, may be mathematically infinite or not. There is nothing in the concept itself of the straight line as developed so far to settle this question. Consequently the assumption made will be an independant one amounting to an axiom. Three assumptions are open to our choice. We may assume that the straight line, as it were, re-enters upon itself, being finite though unbounded. (We may consider, however, bounds in this manner: If A and B be the descriptive bounds of a zero-segment, they are also the bounds of the complete straight line; but since any two such descriptive points may be chosen, no determinate bounds to the complete straight line can be assigned. The next assumption is that if we add non-zero segments one to the other in the straight line, letting the process "tend to infinity" in some manner we should arrive at a final definite bounding point Ω on the straight line. If this process be carried out in the negative direction we should arrive in like manner at the corresponding negative bounding point ω on the straight line. There may be more than one point like Ω and correspondingly more than one like ω according to the process of tending to infinity; i. e. there may be different "points at infinity" corresponding to the

notion of orders of infinity. Or we may consider Ω and ω to be one and the same point so that the straight line re-enters into itself at infinity, having a single bound at infinity called "the point at infinity on the straight line." This last is the Euclidean assumption. Thus a straight line may have no point at infinity, or one point at infinity, or two or more points at infinity. These different assumptions lead to different developments called Elliptic, Euclidean or Parabolic, and Hyperbolic geometries respectively. The complete straight line may be finite in length according to the first concept or infinite according to the other two concepts.

21·4. As the straight line is descriptive of the elemental plane we should have in like manner the three necessarily corresponding assumptions: (1) the plane is finite in area but unbounded, re-entering upon itself, (2) it is infinite re-entering upon itself at infinity, or with a single bound at infinity, (3) it is infinite having two or more bounds, which are straight lines, corresponding to different orders of infinity, and not re-entering upon itself. In the same manner there are three corresponding assumptions for space: (1) space is finite in volume but unbounded, re entering upon itself, (2) it is infinite re-entering upon itself and with one bound at infinity, (3) it is infinite having bounds, which are planes, corresponding to different orders of infinity.

21·5. If we assume that all straight lines must be of finite length, i. e. if we deny to the straight line the applicability of the concept of infinite magnitude, then each straight line is finite but unbounded: so also is a plane and so also space. Motion along the straight line will bring one back to the starting point and any two straight lines necessarily meet in finite space. If on the other hand we admit one point at infinity then two straight lines may not necessarily meet in finite space, but may meet at infinity. If we admit more than one point at infinity on the straight line l then we may have several straight lines through the point A, outside l, meeting l at infinity.

22·0. **Parallelism**—An elemental straight line is determined by two descriptive points. These may be two points A and B, or the two infinite points Ω and ω, corresponding to plus infinity and to minus infinity; or the point ω and the finite point O, or the point Ω and the point O. These points may also be considered as elements with the straight line as their common descriptive property. If P is an elemental point outside l, then the descriptive straight line common to P and Ω is called a "parallel" to l; so also is the descriptive straight line common to P and ω. If we accept Ω and ω as one and the same point, that is assume only one point at infinity on a straight line, then we get *Playfair's axiom*, viz. that there is only one parallel to a straight line through an external point. This is the Euclidean concept.

22·1. **The Euclidean Axiom of Unique Determination.**—We have said that in a plane two descriptive straight lines determine one elemental point, and two elemental straight lines have one descriptive point in common; also that two descriptive points determine one elemental straight line and two

elemental points have one descriptive straight line in common. (See In 1. 15). If we assume that *there is ONE AND ONLY ONE element satisfying the two descriptive properties, and ONE AND ONLY ONE descriptive property common to two elements*, then we have a Euclidean Axiom, and Playfair's Axiom in the last paragraph must follow, for the points P and Ω determine *ONE AND ONLY ONE* straight line. This axiom may be stated thus: (1) two straight lines intersect in one and only one point, or two straight lines cannot enclose a space. This is equivalent to two points determine one and only one straight line. (2) two planes determine one and only one straight line, or the equivalent statement: two straight lines with one point in common determine one and only one plane.

22·2. If, however, this axiom be not taken for granted, we may admit two points of intersection of two straight lines; or we may admit the case where two straight lines in a plane may not intersect at all. In the former case to determine which point of intersection of two straight lines is in consideration, we must have a criterion for distinguishing between them. This is provided by the concept of length.

22·3. These assumptions do not in any way contradict the other axioms but they lead to different developments. The notion of "curvature" is of importance in this connection.

23·0. **The Line at Infinity.**—Euclidean axioms lead to these results: the descriptive bound of a complete plane is a straight line, called the "Line at Infinity", on which all points at infinity on straight lines in the plane lie; or the plane re-enters upon itself at infinity along a straight line. Also the descriptive bound of the whole of space is a plane, called the "Plane at Infinity", on which lie all the straight lines at infinity of all planes in space; or space re-enters upon itself at infinity along a plane.

24·0. To sum up: The homogeneity of spatial elements made use of in Descriptive Geometry is supplemented by the Axiom of Length as an intrinsic property of the straight line segment. Another correspondence or method of identification was rendered possible, viz in magnitude, by means of this axiom together with the Axiom of Free Mobility which rendered superposition possible and provided the basis for the equation of magnitudes. The introduction of magnitude in linear, areal and volumetric spatial figures, together with the postulate of three types of elements, led to the Axiom of Three Dimensions. These are the axioms of Metrical Geometry. One other axiom gives us the whole foundation of Euclidean Geometry.

ON DESMIC TETRAHEDRA

BY

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1. Introduction. (i) The three tetrahedra XX′UU′, YY′VV′, ZZ′WW′ * possess the following properties: (*a*) A line joining any vertex of one to any vertex of another passes through a vertex of the third; (*b*) The line of intersection of any face of one with any face of another lies in a face of the third; (*c*) Any two of these tetrahedra correspond to each other in four harmonic homologies whose centres and planes of homology are the vertices and the respectively opposite faces of the third tetrahedron. † The tetrahedra ABCD, MM′M″M‴, belong to a second group of desmic tetrahedra, the third member of which has for its vertices the point M_a harmonically separated by the vertex A and the face BCD of ABCD, and the three analogous points M_b, M_c, M_d. The two groups of desmic tetrahedra are said to be "associated."

(ii) The twelve centres of similitude of four given spheres taken in pairs, are the vertices of tetrahedra belonging to a desmic system. The tetrahedron having for its vertices the centres of the given spheres belongs to the associated desmic group. ‡

The object of this paper is to establish the converse of this last proposition. We shall begin by proving the following.

2. Lemma. *If a sphere is coaxal with two given spheres and divides their line of centres harmonically, it coincides with the sphere of similitude of the given spheres.*

The variable sphere having for diameter a segment which divides the line of centres AB of the two given spheres (A), (B), harmonically belongs to a coaxal system having the centres A, B, of the given spheres for limiting points. On the other hand the spheres (A), (B), determine a second coaxal system. Now these two systems can have only one common sphere, and the sphere of similitude of (A) and (B) is such a common sphere of the two systems, hence the proposition.

* See *The Mathematics Student*, Vol. II, 1934, pp. 107-111.

† For bibliographical references see R. W. H. T. Hudson, *Kummer's Quartic Surface*, Cambridge, 1905.

‡ J. L. Coolidge. *A Treatise on the Circle and the Sphere*, p. 231, Oxford, 1916.

3. Theorem. *The twelve vertices of a desmic group of three tetrahedra may always be considered to be the centres of similitude of four spheres, taken in pairs.*

Consider the six spheres (UU′), (VV′), ... (ZZ′), having for diameters the six segments UU′, VV′, ... ZZ′ (1) Let (D) be a sphere, of arbitrary radius, having for centre the point D, and let (A) be the sphere having A for centre and coaxal with the spheres (D) and (UU′). Since the points U, U′, are harmonic with respect to D, A, and the sphere (UU′) is coaxal with (A) and (D), the sphere (UU′) is the sphere of similitude of (A) and (D), (2) and therefore the points U, U′, are the centres of similitude of (A) and (D).

If (B), (C), are the spheres having for centres the points B, C, and respectively coaxal with the pairs of spheres (D), (VV′); (D), (WW′), we may show in a similar way that (VV′), (WW′), are, respectively, the spheres of similitude of the pairs of spheres (D), (B); (D), (C), and that the pairs of points V, V′; W, W′, are the centres of similitude of these pairs of spheres.

The three points U, V, Z, are coplanar, for they lie in the two planes DAB and σ. Now U, V, are centres of similitude of the pairs of spheres (D), (A); (D), (B), hence Z is a centre of similitude of the spheres (A), (B), and Z′ is their second centre of similitude. Likewise for the points X, X′; Y, Y′. The proposition is thus proved.

It should be observed that the radius of the sphere (D) was taken arbitrarily, hence the radii of the four spheres are only determined in their ratios, but not in absolute length. Moreover, any one of the three tetrahedra ABCD, MM′M″M‴, $M_aM_bM_cM_d$, of the associated group may be made to play the rôle the preceding proof assigns to the tetrahedron ABCD.

4. The six spheres of similitude of four given spheres taken in pairs belong to the same coaxal net of spheres, * hence their centres are coplanar. Thus:

The mid-points of the six segments which the twelve vertices of a desmic group of tetrahedra determine on the six edges of a tetrahedron of the associated group are coplanar.

5. Applications. (i) The six mid-points of the edges of a tetrahedron and the six points at infinity on these edges are twelve

* Coolidge, *ibid*, p. 244.

points forming a desmic system. They may be considered to be the centres of similitude of four equal spheres, of arbitrary radius, having for centres the vertices of the tetrahedron.

(ii) The twelve bisecting planes of the six dihedral angles of a tetrahedron ABCD determine on the respectively opposite edges twelve points which are the vertices of a desmic group of tetrahedra. These twelve points are the centres of similitude of the four spheres, taken in pairs, which have their centres at the vertices A, B, C, D, and have their radii proportional to the respective altitudes of the tetrahedron ABCD.*

(iii) The twelve pairs of bisectors of the twelve face angles of a tetrahedron ABCD determine on the respectively opposite edge twenty-four points. But if the tetrahedron is isodynamic,† i.e., AB.CD = AC.BD = AD.BC, these twenty-four points coincide in pairs and thus reduce to twelve.‡ It may readily be shown that these twelve points are the centres of similitude of four spheres, taken in pairs, whose centres coincide with the vertices of ABCD and the squares of whose radii are proportional to the products of the three edges passing through the respective vertices of the tetrahedron.

* Nathan Altshiller-Court, On the Apollonian spheres of a tetrahedron. *The Tohoku Mathematical Journal*, vol. 39, part 2 (1934), p. 264.

† J. Neuberg, Memoire sur le tétraèdre. *Académie Royale de Belgique*, vol. 37 (1884); supplement to *Mathesis*, vol. 5 (1885). See also *Journal of the Indian Math. Soc.*, vol. 5 (1913), p. 35, Q. 404; p. 115, Q. 436; and pp. 216-219.

‡ *Mathesis* 1884, p. 68, Question 253. *Educational Times, Reprints*, vol. 13 (1908), p. 83, Q. 16265.

GLEANINGS

Mathematics	*Religion*
1. The Infinite exists.	1. God exists.
2. Eternal laws exist.	2. Eternal laws exist.
3. The laws relating to finite magnitudes do not hold respecting the infinitely large or the infinitely small.	3. God's laws are so different from ours as to be absolutely non-understandable by us.
4. The existence of hyperspace with gradations, is entirely reasonable.	4. The existence of a heaven with gradations, is entirely reasonable.
5. No factor is ever lost.	5. The soul is eternal.
6. Time may be a closed curve.	6. God looks at time as a whole.
7. Time may be a fourth dimension.	7. In the next world, the direction of time may actually be seen.
8. Positive infinity may physically coincide with negative infinity, if lines curve through four-space.	8. In God's sight the infinite past and the infinite future are the same.

NOTES AND DISCUSSIONS

Note on the expression of a biquadrate as a sum of three biquadrates

It has not yet been proved that it is impossible to exhibit a biquadrate as a sum of three biquadrates, (this being a special case of a conjecture of Euler that it is impossible to exhibit a perfect n^{th} power as a sum of m n^{th} powers where $m < n$).

Dickson, in his "History of Theory of Numbers",* gives the impression that the impossibility of

$$x^4 + y^4 + z^4 = u^4 \qquad \ldots \quad (1)$$

was proved by Werebrusow. Werebrusow† starts with the identity (of Euler)

$$(a^2 + b^2 + c^2 + d^2)^2 = (a^2 + b^2 - c^2 - d^2)^2 + (2ac + 2bd)^2 + (2ad - 2bc)^2 \qquad \ldots \quad (2)$$

Making three of the terms in this identity perfect biquadrates, he arrives at

$$(3 + 4t^2)^4 = (-1 + 4t^2)^4 + (2 + 4t)^4 + 16[2(2t - 1)(2t^2 + 2t + 1)]^2$$

and notes that the last term cannot be made a biquadrate. But by

* Vol II, 648.

† *L'intermédiaire des Math.* 21, 1914, 161.

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9. A Flatlander has enough of the third dimension in his being to give him some feeling of that dimension; and so this may explain the fact that we have some feeling of the fourth dimension.

9. The human soul has enough of the divine within it to have some feeling of the reality of divinity and of the world beyond.

10. Mathematics is a vast storehouse of the discoveries of the human intellect. We cannot afford to discard this material.

10. Religion is a vast storehouse of the discoveries of the human spirit. We cannot afford to discard this material.

11. It is not necessary that the solution of a problem, by limited means—say the trisection of an angle—should be found in order that we may feel certain that the problem can be solved by some means.

11. It is not necessary that the solution of the problem of religion, by our limited human means, should be found in order that we may feel certain that the problem can be solved by some means

12. Every term in an infinite sequence is in a small way a part of infinity.

12. Lucretius spoke wisely when he said "Every one is in a small way the image of God."

D. E. SMITH in his Essay on Religio Mathematici, *Poetry of Mathematics and other Essays.*

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giving any special value to t, e.g. $t=1$, we see that this equation is incorrect. What was probably meant is the identity

$$(3+4t^2)^4=(-1+4t^2)^4+(2+4t)^4+16[2(2t-1)(2t^2+t+1)]^2,$$

which follows from (2) by putting $a=2$, $b=1+4t^2$, $c=4t$ and $d=2$. (In our equation three of the terms tally with that of Werebrusow.) In this form of the equation it does not seem easy to prove that the last term cannot be made a biquadrate.

Since Euler's identity gives the most general solution* of $x^2+y^2+z^2=u^2$, the conjecture that (1) is impossible is equivalent to the conjecture that not all terms in (2) can be rendered biquadrates Still, contrary to the impression given by Werebrusow and Dickson, it seems unlikely that the impossibility of (1) can be proved in this manner.

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Lecture Notes

I. *The Pole and Polar.*

The usual method of deriving the polar of a point with respect to a conic involves first the determination of the chord of contact of tangents, when the polar is regarded as the locus of the intersection of tangents at the ends of chords through the given point. The following is a more direct method available for any system of homogeneous co-ordinates.

Let $P(x_1, y_1, z_1)$ be the given point and

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

the given conic. If $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ be the intersections of any line through P and the conic S, we have

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0, \quad \text{and therefore} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0$$

i.e.

$$\frac{1}{8}\begin{vmatrix} \frac{\partial S}{\partial x_1} & \frac{\partial S}{\partial y_1} & \frac{\partial S}{\partial z_1} \\ \frac{\partial S}{\partial x_2} & \frac{\partial S}{\partial y_2} & \frac{\partial S}{\partial z_2} \\ \frac{\partial S}{\partial x_3} & \frac{\partial S}{\partial y_3} & \frac{\partial S}{\partial z_3} \end{vmatrix}=0,$$

by the usual rules for the multiplication of two determinants.

* Carmichael, "Diophantine Analysis", Page 39.

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This shows that the tangents at Q, and R meet on

$$x\,\frac{\partial S}{\partial x_1} + y\,\frac{\partial S}{\partial y_1} + z\,\frac{\partial S}{\partial z_1} = 0 \qquad \dots \quad (1)$$

which is the equation of the required polar.

Re-interpreting the symbols (x, y, z) as co-ordinates of a line and suitably modifying the phraseology, the above algebra leads to the equation of the pole of the line (x_1, y_1, z_1) in the form (1). Thus, the pole of the line $xx_1+yy_1+zz_1=0$ w.r.t. a conic $\Sigma=0$ in line co-ordinates is $\partial\Sigma/\partial x_1$, $\partial\Sigma/\partial y_1$, $\partial\Sigma/\partial z_1$.

II. *Foci and Axes.*

Let $I\,(x_1+ix_2,\ y_1+iy_2,\ z_1+iz_2)$, $J\,(x_1-ix_2,\ y_1-iy_2,\ z_1-iz_2)$ be the circular points at infinity in any system of homogeneous co-ordinates.

Let $S_1\equiv ax_1^2+\cdots+2fy_1z_1+\cdots,\qquad S_2\equiv ax_2^2+\cdots+2fy_2z_2+\cdots$

$T_1\equiv axx_1+\cdots+f(yz_1+y_1z)+\cdots,\quad T_2\equiv axx_2+\cdots+f(yz_2+y_2z)+\cdots$

$T_{12}\equiv ax_1x_2+\cdots+f(y_1z_2+y_2z_1)+\cdots$

The polars of I, J w.r.t. the conic $S\equiv ax^2+\cdots+2fyz+\cdots=0$ are $T_1+iT_2=0$ and $T_1-iT_2=0$. The foci, being the intersections of tangent pairs from I, J, are given by

$$S(S_1-S_2+2iT_{12}) = (T_1+iT_2)^2;\quad S(S_1-S_2-2iT_{12}) = (T_1-iT_2)^2$$

which are equivalent to

$$\frac{T_1^2-T_2^2}{S_1-S_2} = \frac{T_1T_2}{T_{12}} = S. \qquad \dots \quad (1)$$

By the usual argument, the equation of the axes is

$$\frac{T_1^2-T_2^2}{S_1-S_2} = \frac{T_1T_2}{T_{12}}. \qquad \dots \quad (2)$$

The above results are a generalisation of the usual Cartesian equations obtained by putting $(x_1, y_1, z_1)\equiv(1, 0, 0)$ and $(x_2, y_2, z_2)\equiv(0, 1, 0)$.

The appropriate further simplification for any particular system of homogeneous co-ordinates is left to the reader.

III. *Director-circle and Directrices.*

The Director-circle and the directrices of a conic $S=0$ can both be obtained from the same investigation, by noting that the common chords of $S=0$ and its director-circle are

(i) The two pairs of directrices corresponding to the real and the imaginary foci and (ii) the polars of the circular points at infinity with respect to $S=0$. (*Vide* D. M. Y. Sommerville, *Analytical Conics*,

P. 291). Thus, the director-circle and the directrices belong to the linear system of conics determined by $S=0$ and $(T_1+iT_2)(T_1-iT_2)=0$, (using the notation of II above.)

Their equations are, therefore, of the form

$$\lambda\ S+T_1^2+T_2^2=0 \qquad \dots \quad (1)$$

To get 'λ' for the director-circle, use the condition that (1) passes through the circular points. We thus find $\lambda=-(S_1+S_2)$.

Hence the director-circle is given by

$$S(S_1+S_2)=T_1^2+T_2^2 \qquad \dots \quad (2)$$

To get 'λ' for the directrices, we have to use the condition that (1) represents a pair of lines, rejecting the solution $\lambda=0$. If we put $S'\equiv T_1^2+T_2^2$, λ must satisfy the quadratic equation $\lambda^2\Delta+\lambda\theta+\theta'=0$ in the usual invariants-notation.

Let us try to simplify θ and θ'. Writing

$$\xi_r\equiv ax_r+hy_r+gz_r,\quad \eta_r\equiv hx_r+by_r+fz_r,\quad \zeta_r\equiv gx_r+fy_r+cz_r \quad (r=1,\ 2),$$

we have

$$a'\equiv\xi_1^2+\xi_2^2,\ \dots\dots; \qquad f'\equiv\eta_1\zeta_1+\eta_2\zeta_2;\ \dots$$

$$A'\equiv(\eta_1^2+\eta_2^2)(\zeta_1^2+\zeta_2^2)-(\eta_1\zeta_1+\eta_2\zeta_2)^2\equiv(\eta_1\zeta_2-\eta_2\zeta_1)^2,\dots\dots$$

$$F'\equiv(\zeta_1\xi_1+\zeta_2\xi_2)(\xi_1\eta_1+\xi_2\eta_2)-(\xi_1^2+\xi_2^2)(\eta_1\zeta_1+\eta_2\zeta_2)$$
$$\equiv(\zeta_1\xi_2-\zeta_2\xi_1)(\xi_1\eta_2-\xi_2\eta_1),\dots\dots$$

and therefore,

$$\theta=a'A+\ \dots\ +2f'F+\ \dots$$

$$=(A\xi_1^2+\ \dots\ +2F\eta_1\zeta_1+\ \dots)+(A\xi_2^2+\ \dots\ +2F\eta_2\zeta_2+\ \dots)$$

$$=-\begin{vmatrix} a & h & g & \xi_1 \\ h & b & f & \eta_1 \\ g & f & c & \zeta_1 \\ \xi_1 & \eta_1 & \zeta_1 & 0 \end{vmatrix} - \begin{vmatrix} a & h & g & \xi_2 \\ h & b & f & \eta_2 \\ g & f & c & \zeta_2 \\ \xi_2 & \eta_2 & \zeta_2 & 0 \end{vmatrix}$$

$$=-\begin{vmatrix} a & h & g & 0 \\ h & b & f & 0 \\ g & f & c & 0 \\ \xi_1 & \eta_1 & \zeta_1-S_1 \end{vmatrix} - \begin{vmatrix} a & h & g & 0 \\ h & b & f & 0 \\ g & f & c & 0 \\ \xi_2 & n_2 & \zeta_2-S_2 \end{vmatrix}^{*} =(S_1+S_2)\Delta.$$

* Transform the last column of the first determinant by adding to it the first three columns multiplied respectively by $-x_1, -y_1, -z_1$. Similarly for the second determinant use the multipliers $-x_2, -y_2, -z_2$.

Again, $\theta' = aA' + \ldots + 2f\,F' + \ldots$

$$= \begin{vmatrix} a & h & g & \xi_1 & \xi_2 \\ h & b & f & \eta_1 & \eta_2 \\ g & f & c & \vartheta_1 & \vartheta_2 \\ \xi_1 & \eta_1 & \zeta_1 & 0 & 0 \\ \xi_2 & \eta_2 & \zeta_2 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a & h & g & 0 & 0 \\ h & b & f & 0 & 0 \\ g & f & c & 0 & 0 \\ \xi_1 & \eta_1 & \vartheta_1 - S_1 & -T_{12} \\ \xi_2 & \eta_2 & \vartheta_2 - T_{12} - S_2 \end{vmatrix}$$

$$= \Delta(S_1S_2 - T_{12}^2).$$

Thus, λ for the directrices satisfies the quadratic equation

$$\lambda^2 + \lambda(S_1 + S_2) + S_1S_2 - T_{12}^2 = 0. \qquad \ldots \quad (3)$$

IV. *Asymptotes.*

An asymptote is *parallel to the polar of any point on it.* Hence, if $px + qy + rz = 0$ is the equation of the line at infinity, and (x_1, y_1, z_1) any point on an asymptote, $lx + my + nz = 0$ of the conic $S = 0$ (in point-co-ordinates) or $\sum = 0$ (in line-co-ordinates) we have

$$\begin{vmatrix} \frac{\partial S}{\partial x_1} & \frac{\partial S}{\partial x_2} & \frac{\partial S}{\partial x_3} \\ l & m & n \\ p & q & r \end{vmatrix} = 0$$

Combining (1) with two other well-known relations

$$lx_1 + my_1 + nz_1 = 0\,; \qquad l\frac{\partial \Sigma}{\partial p} + m\frac{\partial \Sigma}{\partial q} + n\frac{\partial \Sigma}{\partial r} = 0.$$

and eliminating (l, m, n), we get the locus of (x_1, y_1, z_1), i.e. the equation of the asymptotes in the form

$$\begin{vmatrix} q\frac{\partial S}{\partial z} - r\frac{\partial S}{\partial y} & r\frac{\partial S}{\partial x} - p\frac{\partial S}{\partial z} & p\frac{\partial S}{\partial y} - q\frac{\partial S}{\partial x} \\ \frac{\partial \Sigma}{\partial p} & \frac{\partial \Sigma}{\partial q} & \frac{\partial \Sigma}{\partial r} \\ x & y & z \end{vmatrix} = 0$$

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V. *Bisectors of the angles between two given straight lines.*

In discussing the bisectors of the angles between two straight lines, it is customary to distinguish the bisector of the angle in which the origin lies; practically no attention is paid to determine whether it bisects the acute or the obtuse angle.

(a) Now the bisector of the acute angle makes with either of the lines an angle θ less than 45° or greater than 135° so that $|\tan\theta| < 1$ and the bisector of the obtuse angle makes with either of the lines an angle ϕ greater than 45° or less than 135°, so that $|\tan\phi| > 1$. This gives a test for determining which angle each bisects.

Let the given lines be: $ax+by+c=0$ (i) $a'x+b'y+c'=0$ (ii)

The equations of the bisectors are

$$\frac{ax+by+c}{\lambda} \mp \frac{a'x+b'y+c'}{\lambda'} = 0 \text{ where } \lambda = \sqrt{a^2+b^2} \text{ and } \lambda' = \sqrt{a'^2+b'^2}$$

i.e.
$$(a\lambda' - a'\lambda)x + (b\lambda' - b'\lambda)y + c\lambda' - c'\lambda = 0 \qquad \text{(iii)}$$

$$(a\lambda' + a'\lambda)x + (b\lambda' + b'\lambda)y + c\lambda' + c'\lambda = 0 \qquad \text{(iv)}$$

The angles θ, ϕ between the bisectors (iii) and (iv) and one of the given straight lines say (i) are given by

$$\tan\theta = \frac{ab'-a'b}{\lambda\lambda' - aa' - bb'}; \quad \tan\phi = -\frac{ab'-a'b}{\lambda\lambda' + aa' + bb'}.$$

It can be shown that if $|\tan\theta| < 1$ then $|\tan\phi| > 1$ since their product is unity.

Hence if $|ab'-a'b| < |\lambda\lambda' - aa' - bb'|$ the equation (iii) represents the bisector of the acute angle; otherwise it will bisect the obtuse angle.

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SOLUTIONS TO QUESTIONS

Question 1647

(A. A. KRISHNASWAMY AYYANGAR):—If confocal conics reciprocate into coaxal circles with respect to a conic, then this conic is either a circle or a rectangular hyperbola with the same centre.

Solution by A. Ranganatha Rao.

Let O be the centre of the base conic $\sum$ of reciprocation, and i, j the lines through O, which are polars of the circular points I, J with respect to $\sum$. Since the coaxal circles all pass through I, J, the confocal conics all touch i, j. But the only tangents common to the confocal system are the two pairs of circular lines through the common foci. Hence i, j are circular lines and O a common focus. There are only two possibilities, namely (1) i passes through I and j through J, in which case $\sum$ is a circle with centre O or (2) i passes through J and j through I, in which case I, J are conjugate points with respect to $\sum$ and hence $\sum$ is a rectangular hyperbola with centre O.

Also solved by M. K. Hariharan and T. K. Raghavachari; and analytically by. S. P. Ranganathachar.

Question 1657

(K. J. SANJANA.) Obtain the numerical value of the expression $\cos\frac{5\pi}{37} + \cos\frac{13\pi}{37} + \cos\frac{15\pi}{57} + \cos\frac{17\pi}{37} + \cos\frac{19\pi}{37} + \cos\frac{33\pi}{37} + \cos\frac{29\pi}{37} + \cos\frac{31\pi}{37} + \cos\frac{35\pi}{37}$.

Solution by N. K. Narasimhamurti.

Let $a = \cos\frac{2\pi}{37} + i\sin\frac{2\pi}{37}$ and let $p = \sum a^n$ where n takes the values

1, 4, 16, 27, 34, 25, 26, 30, 9, 36, 33, 21, 10, 3, 12, 11, 7, 28

which are congruent to successive powers of 4 modulo 37; and let $q = \sum a^n$ for the other values of n. Then $p+q = -1$ and $pq = 9(p+q) = -9$ so that p, q are roots of $x^2 + x - 9 = 0$ i.e. they are $\frac{-1 \pm \sqrt{37}}{2}$.

Also $p = (a + a^{36}) + (a^4 + a^{33}) + \cdots$

$$= (a + a^{-1}) + (a^4 + a^{-4}) + \cdots \text{ to 9 terms}$$

$$= 2\sum_r \cos\frac{2r\pi}{37} \text{ where } r = 1, 4, 16, 27, 34, 25, 26, 30, 9.$$

The expression given in the question is $-\frac{1}{2}p$ since

$$\cos\frac{2\pi}{37} = -\cos\frac{35\pi}{37},\ \cos\frac{8\pi}{37} = -\cos\frac{29\pi}{37}\ \text{etc.}$$

Hence its value is $(1 \pm \sqrt{37})/4$. The ambiguity is removed by noting that the positive terms (the first four in the given expression) is less than the numerical value of the terms which follow. Hence the expression is negative and its value is $(1-\sqrt{37})/4$.

Also solved by N. Maitra and T. Raghavachari

Question 1661

(K. J. SANJANA, M.A.).—In a triangle ABC each of the angles at B and C is double of the angle at A. Prove geometrically that :—

(1) The straight lines joining the point C to the circumcentre and in-centre of the triangle meet the opposite side AB in points which are isotomically conjugate.

(2) Four times the sum of the squares on the perpendiculars from A on BC and from B on CA are equal to five times the square on AB.

Solution by N. K. Narasimhamurthy.

(1) Let S, I, be the circum-, and in-centres of ABC, and let CS, CI meet AB at X, Y respectively. CY bisects $\angle$ ACB. Therefore

$$\angle BCY = 36^\circ = \angle ACY.$$

$\therefore\quad AY = CY = BC.$

$\angle ACS = \angle CAS = 18^\circ \quad \therefore\ \angle BCX = 54^\circ = \angle BXC.$ Hence $BX = BC$.

$\therefore\quad AY = BX$ and $BY = AX$. Hence the result.

(2) Let AD, BE be respectively perpendicular to BC, AC.

Take EF in EA equal to EC. Join BF.

Then $BC = BF = AF$ and each is of the length of the major segment of the medial section of AC. Hence $BC^2 = AF^2 = AC.\ CF$. Now,

$$\begin{aligned}
4(AD^2 + BE^2.) &= (4AB^2 - BC^2) + (4BC^2 - CF^2.) \\
&= 4AB^2 + BC^2 + 2AC.\ CF - CF^2. \\
&= 4AB^2 + BC^2 + 2(BC + CF)CF - CF^2. \\
&= 4AB^2 + BC^2 + 2BC.\ CF + CF^2 = 4AB^2 + (BC + CF)^2. \\
&= 4AB^2 + AC^2 = 5AB^2.
\end{aligned}$$

Also solved by K. Subramanyam, N. Maitra, A. L. Shaikh and T. K. Raghavachari.

Question 1665

(K. RANGASWAMY).—The chords PG, PH through any point P of the ortho-centroidal circle of the triangle ABC bisect the angles between one of the tangents through P to the circumcircle and one to the nine-point circle, where G and H are the centroid and the orthocentre of ABC.

Solution by Nathan Altshiller-Court

The points G, H, divide the line of centers of the circumcircle (O) and the nine-point circle (N) of ABC in the ratio of the radii of these circles, hence G, H, are the centres of similitude of these two circles and the ortho-centroidal circle having GH for diameter is the circle of similitude of (O) and (N). Since G, H divide the segment ON harmonically, the pencil P(GH, ON) is harmonic; the lines PG, PH being orthogonal, they are the bisectors of the angles formed by PO and PN. On the other hand the two circles (O). (N) subtend equal angles at any point P of their circle of similitude (GH). Thus the stated proposition follows immediately, as is readily seen from a figure.

It may be observed that the proposition is valid for any point of the circle of similitude of any two circles.

Also solved by N. K. Narasimhamurti, S. Narayanan, and A. Ranganatha Rao.

Question 1667

(A. A. KRISHNASWAMI AYYANGAR):—Prove that

$$\int_0^x e^{-\frac{1}{2}x^2}dx > \sqrt{(\pi/2)}\,(1-e^{-x^2}), \qquad (x>o).$$

Remarks by C. N. Srinivasiengar.

The result is incorrect, except in a small range of positive values for x. This fact can be verified from well-known statistical considerations. For the frequency distribution $y=\frac{\mathrm{N}}{(2\pi)^{\frac{1}{2}}\sigma}e^{-x^2/2\sigma^2}$, the probability of an "error" between $-k\sigma$ and $+k\sigma$ is given by $\frac{2}{(2\pi)^{\frac{1}{2}}}\int_0^k e^{-x^2/2}\,dx$. Using the probability integral table, the values of this expression when $k=2$, and when $k=3$ are respectively 0·9545 and 0·9973. But

$1 - e^{-k^2}$ for these values give 0·9817 and 0·9999. The result is therefore incorrect. Let

$$f(x) = \int_0^x e^{-\frac{1}{2}x^2}dx + \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-x^2}; \qquad f'(x) = e^{-\frac{1}{2}x^2}\left\{1 - (2\pi)^{\frac{1}{2}}\, xe^{-x^2/2}\right\}.$$

Let $\phi(x) = xe^{-x^2/2}$. It will be verified that $\phi(x)$ is maximum when $x = 1$, and increases in $(0, 1)$, and decreases in $(1, \infty)$. There are thus two values of x, call them α and β, one in each of these intervals, for which $xe^{-x^2/2} = (2\pi)^{-\frac{1}{2}}$. It will be verified that α lies between $\frac{1}{4}$ and $\frac{1}{2}$, while β lies between $1\frac{1}{2}$ and 2.

Hence $f^1(x)$ is positive in (o, α), negative in (α, β), and positive in (β, ∞). $f(x)$ is thus an increasing function in (β, ∞), and the *final* value is $\int_0^\infty e^{-\frac{1}{2}x^2}dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$. Hence $f(x) < \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$ in (β, γ) where γ is any number greater than β:

[The solution by Mr. A. L. Shaikh published on p. 123, *Mathematics Student*, Vol. II contains an accidental mistake in the seventh line of that page. On changing x to $x\sqrt{2}$ in $\int_0^x e^{-\frac{1}{2}x^2}dx$, the limits of integration become 0, $a/\sqrt{2}$ and not $0, a\sqrt{2}$ as written there. Mr. Shaikh's method, worked out with this correction, gives $\int_0^x e^{-\frac{1}{2}x^2}dx > \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left((1 - e^{-\frac{1}{4}x^2}\right)$, which gives only a poor approximation to the probability integral when x receives values like 2 or 3.]

[The mistake in Mr. A. L. Shaik's solution has also been pointed out by the proposer.—*Ed.*]

Remarks by K. Venkatachaliengar

The inequality given in the above question is equivalent to

$$\psi(a) = \int_0^a e^{-x^2}dx > \frac{\sqrt{\pi}}{2}\left(1 - e^{-2a^2}\right) \qquad \dots \quad (1)$$

This is not true and the best possible inequality of the type is

$$\int_0^a e^{-x^2}dx > \frac{\sqrt{\pi}}{2}\left(1 - e^{-a^2}\right) \qquad \dots \quad (2)$$

The result that we are going to prove is a generalisation of this inequality. The result is

$$\int_0^a e^{-x}x^{t-1}dx > \Gamma(t)\,(1-e^{-a}) \quad \text{for } 0<t<1,$$

(and the reverse for $t>1$), and an inequality of the type

$$\int_0^a e^{-x}x^{t-1}dx > \Gamma(t)[1-e^{-(\lambda+1)a}] \text{ where } \lambda>0,$$

is not true. (In case $t>1$, the reverse sign should be introduced in the inequality and $\lambda<0$).

We will only consider the case $0<t<1$, as the proof for $t>1$ is exactly similar.

Consider the function, $F(a) = \int_0^a e^{-x}x^{t-1} - \Gamma(t)\,(1-e^{-a})$. We obviously have $F(0)=F(\infty)=0$, and $F'(a)=e^{-a}(a^{t-1}-\Gamma(t).)$ $\therefore$ $F'(a)$ is positive up to a certain stage and then it is negative throughout as a increases to ∞. From this we deduce that $F(a)$ increases up to a certain stage and then decreases monotonically to zero. Hence $F(a)$ is positive for all positive values of a. Replacing x by x^2, we get (2). In order to prove the latter part of the inequality consider the function

$$F(a) = \int_0^a e^{-x}x^{t-1}dx - \Gamma(t)\,[1-e^{-(\lambda+1)a}], \quad \text{We have } F(0)=F(\infty)=0,$$

and $F'(a)=e^{-a}[a^{t-1}-\Gamma(t)\,(\lambda+1)e^{-\lambda a}]$. From this it is clear that $F'(a)$ is positive for sufficiently small values of a and for sufficiently large values of a (depending on λ which is greater than zero.) Hence $F(a)$ increases from zero for small values of a and it monotonically increases to zero after a certain stage, as $a\to\infty$, and therefore $F(a)$ should be negative for sufficiently large values of a.

Question 1669

(HANSRAJ GUPTA).—If $|x|<1$

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)\ldots\ldots} = 1 + \sum_{s=1}^{\infty} \frac{x^{2s}}{(1-x)(1-x^2)\ldots\ldots(1-x^s)}.$$

Solution by S. S. Pillai

Let $$S_n = 1 + \sum_{s=1}^{n-1} \frac{x^{2s}}{(1-x)(1-x^2)\ldots\ldots(1-x^s)}$$

Let us assume that

$$S_n = \frac{1-x+x^n}{(1-x)(1-x^2)\ldots\ldots(1-x^{n-1})} \qquad \ldots(1)$$

for a certain value of n. Then

$$S_{n+1} = S_n + \frac{x^{2n}}{(1-x)\ldots(1-x^n)}$$
$$= \frac{(1-x+x^n)(1-x^n)+x^{2n}}{(1-x)\ldots\ldots(1-x^n)}$$
$$= \frac{1-x+x^{n+1}}{(1-x)\ldots\ldots(1-x^n)}$$

But $\qquad S_2 = 1 + \frac{x^2}{1-x} = \frac{1-x+x^2}{1-x}$

Hence, by induction, we get that S_n is of the form (1).

So, since $|x|<|$, passing to the limit $n \to \infty$ we have

$$S_\infty = \frac{1-x}{(1-x)(1-x^2)\ldots} = \frac{1}{(1-x^2)(1-x^3)\ldots}$$

Also solved by the Proposer, A. L. Shaikh, A. Ranganatha Rao and N. K. Narasimhamurti.

Question 1670

(B. RAMAMURTI) :—If $f_r(x)$, $r = 1, 2, 3, 4$ be four linearly independent polynomials of the same degree, and J_{rs} denotes the Jacobian of $f_r(x)$ and $f_s(x)$ prove that

$$\Sigma\ [J_{12}(x)\ J_{34}(y) + J_{12}(y)\ J_{34}(x)]$$

has the factor $(x-y)^4$, the summation corresponding to the permutations (12) (34), (23) (14) and (31) (24).

Hence, or otherwise, prove that if r_1, r_2, r_3, r_4 be four distinct integers, the equation $\Sigma \left(x^{r_1+r_2-1} + x^{r_3+r_4-1} \right)(r_1-r_2)(r_3-r_4) = 0$ has unity as a quadruple root.

Solution by A. Ranganatha Rao.

Let us replace the four polynomials by four binary forms a_x^n, b_x^n, c_x^n, d_x^n, in Clebsh-Aronhold symbols. Then, the given expression is, but for a constant factor, equivalent to

$$\Sigma\ (ab)(cd)(a_x^{n-1}\, b_x^{n-1}\, c_y^{n-1}\, d_y^{n-1} + a_y^{n-1}\, b_y^{n-1}\, c_x^{n-1}\, d_x^{n-1}), \qquad \text{(A)}$$

the summation corresponding to the permutations ab, cd ; bc, ad and ca, bd. Using the identity,

$$(ab)(cd) = -(bc)(ad) - (ca)(bd)$$

this expression reduces to

$$(bc)(ad)(b_x^{n-1}\, d_y^{n-1} - b_y^{n-1}\, d_x^{n-1})(c_x^{n-1}\, a_y^{n-1} - c_y^{n-1}\, a_x^{n-1})$$
$$+ (ca)(bd)(a_x^{n-1}\, d_y^{n-1} - a_y^{n-1}\, d_x^{n-1})(c_x^{n-1}\, b_y^{n-1} - c_y^{n-1}\, b_x^{n-1}).$$

Here, $b_x^{n-1}\, d_y^{n-1} - b_y^{n-1}\, d_x^{n-1}$ contains as factor $b_x d_y - b_y d_x$, which is

equal to $(bd)(xy)$. Thus, removing from the above expression the factor $(bc)(ad)(ca)(bd)(xy)^2$, there remains

$$\sum (b_x^{n-i} d_y^{n-i} b_y^{i-2} d_x^{i-2} c_x^{j-2} a_y^{j-2} c_y^{n-j} a_x^{n-j} - a_x^{n-i} d_y^{n-i} a_y^{i-2} d_x^{i-2} c_x^{j-2} b_y^{j-2} c_y^{n-j} b_x^{n-j}), \quad i, j = 2, 3, \ldots n. \qquad \text{(B)}$$

In this sum, the terms corresponding to $i = j$ vanish and those corresponding to $i > j$ may be written

$$\sum b_x^{n-i} d_y^{n-i} b_y^{j-2} d_x^{j-2} c_x^{j-2} a_y^{j-2} c_y^{n-i} a_x^{n-i} (b_y^{i-j} a_x^{i-j} d_x^{i-j} c_y^{i-j} - b_x^{i-j} a_y^{i-j} d_x^{i-j} c_y^{i-j}) \qquad \text{(C)}$$

To this we must add the terms corresponding to $i < j$. But we may interchange i and j in (B) and still assume $i > j$. We, thus, obtain another expression of the type (C) and adding this to (C), we get

$$\sum (b_x^{n-i} d_y^{n-i} b_y^{j-2} d_x^{j-2} c_x^{j-2} a_y^{j-2} c_y^{n-i} a_x^{n-i} (a_x^{i-j} b_y^{i-j} - b_x^{i-j} a_y^{i-j}) \times (c_x^{i-j} d_y^{i-j} - d_x^{i-j} c_y^{i-j})$$

which contains the factor $(ab)(cd)(xy)^2$. Hence the original expression (A) has the factor $(ab)(bc)(ca)(ad)(bd)(cd)(xy)^4$. The factor $(xy)^4$ here corresponds to a factor $(x-y)^4$ of the given expression referred to the polynomials.

In particular, taking $f_i(x) = x^{r_i} x'^{n-r_i}$, $i = 1, 2, 3, 4$, we can directly calculate the given function of the Jacobians, the variable x', introduced to secure homogeneity, being made equal to unity at the end of the calculation. Thus, we see that

$$\sum \left(x^{r_1 + r_2 - 1} y^{r_3 + r_4 - 1} + x^{r_3 + r_4 - 1} y^{r_1 + r_2 - 1} \right) (r_1 - r_2)(r_3 - r_4)$$

has $(x-y)^4$ as a factor. Putting $y = 1$, we get the required result.

Solution by the Proposer.

If we put $x_r = f_r(x)$, we have a parametric representation of a rational space curve R_n and $J^{(x)}_{rs}$ are then the line co-ordinates of the tangent to R at the point 'x'. The above expression equated to zero gives the condition for the intersection of the tangents at the points x and y. y is then a root of the Jacobian of the pencil of n-ics determined by planes through the tangent at x. Since every member of the pencil contains x as a double root, the Jacobian has x as a multiple root of order 4. Hence $(x-y)^4$ is a factor of the above expression. The second result follows by taking the functions to be x^{r_1}, x^{r_2}, x^{r_3} and x^{r_4}.

Question 1671

(NATHAN ALTSHILLER COURT, Oklahoma, U.S.A.):—Consider the variable line on which two fixed spheres determine two harmonic segments. The locus of the mid-points of these segments is a sphere,

Solution by A. Ranganatha Rao

Let the centres of the two spheres be P and Q and let any line cut them in pairs of points A, B and C, D respectively. Let O be the mid-point of, say, AB. Then, is AB if harmonically divided by CD, we have $OC.OD = OB^2 = PB^2 - PO^2$.

But, $OC{\cdot}OD = QO^2 - QC^2$ (power of O with respect to the sphere (Q)).

$\therefore \qquad PO^2 + QO^2 = PB^2 + QC^2 = \text{constant}$

The locus of O is, therefore, a sphere whose centre is the mid-point of PQ and which is coaxial with the given spheres.

Solution by the Proposer.

$OP^2 - PB^2 = -(OQ^2 - QC^2)$. Hence the ratio of the powers of O with respect to the two spheres is constant and equal to -1. Consequently the locus of O is a coaxal sphere. This sphere is called the radical sphere of the two spheres. (J. L. Cooledge, *A Treatise on the Circle and the Sphere*, p. 233).

Also solved by V. Ranganathan.

Question 1675

(A. RANGANATHA RAO) :—A point P and a line l are pole and polar with respect to each of two conics S, S′. If *one* chord AB of S′, which is a tangent to S be such that PA, PB are conjugate lines with respect to S, then the same is true of *all* chords of S′ which are tangents to S.

Solution by T. K. Raghavachari.

Project S into a circle with P as centre. Then S′ projects into a concentric conic. Since PA, PB are conjugate with respect to S, $\angle APB = 90^\circ$. Now reciprocate with respect to S. S′ reciprocates into a concentric conic again, and AB reciprocates into a point (ab) on S such that the tangents from it to the reciprocal of S′ are perpendicular. Hence it is easily seen that S is the director circle of the reciprocal of S′. Therefore in the reciprocal figure, all chords AB of S′ tangent to S subtend a right angle at P. Hence PA, PB are conjugate with respect to S. By projecting back, the above result follows.

Also solved by V. Ranganathan, M. K. Srinivasan and the proposer.

REVIEWS

UPENDRANATH RAY, *Analytical Conics*, S. C. Auddy & Co., Calcutta pp. 996. Price Rs. 5/-

The author has taken enormous pains to present in detail the metrical properties of the straight line, the circle, and the conic, and has embodied in his work the result of years of experience in teaching the subject. The first 90 pages are devoted to the exposition of the principles of analytic geometry and the rest to their application to the investigation of geometric properties of conics. The parabola, and the central conic, take each three chapters, the first dealing with the curve, the second with the "chord properties" and the third with the "tangent properties". Systems of circles, and conics, are studied in detail. Each theoretical topic is followed by copious examples, and every detail is completley worked out—the lengths of most of the segments of lines met with, the equations of the circum-circles of triangles inscribed or circumscribed to a conic, the pedal equations, the pedal curves and the evolutes. Every problem is tackled by taking the Cartesian rectangular system, the oblique system, and the polar system of co ordinates, and the solutions are presented side by side.

But the reader cannot help feeling oppressed by this very exuberance of material and the heavy analytic methods, and is led to doubt whether they have not thrown into the background the geometrical aspect, which is more important. From the point of view of a beginner, the main idea should be to enable him to appreciate the methods of analytic geometry, and develop in him a sense for the right method of attack of a problem on hand. To this end, book works, like the equation of the nine-point circle of the triangle formed by three conormal points on a parabola, which have no geometrical interest might well have been given as questions. A separate treatment of the tangents, after finishing all about chords, does not seem to be a change for the better. A chapter on the general conic, would have rendered unnecessary a detailed working of the case of each conic by identical methods while such vague conclusions like the one on p. 284 that the circle is "therefore completely round as shown in figure" might have been avoided. The fact that the conic is the section of any cone with a circular base and not necessarily the right circular cone may be pointed out.

It is surprising that in a book swelling to thousand pages, and claimed to cover 'the Bachelor's course both Pass and Honours of any Indian University', the system of homogeneous co-ordinates which are specially suited to the introduction of elements at infinity, and the elegant and interesting theory of the invariants of conics, have not been considered at all.

The general get-up and printing is good. The lean symbol for infinity may be replaced by a more robust one, and the symbol $<$ for the angle does not look pleasing to the eye.

On the whole the treatment is very exhaustive, in the limited scope to which it is confined and with the numerous examples at the end of each chapter, it should serve as a good book of reference; considering the labour involved, and the richness of its contents, the price is very moderate.

B. R.

BOOKS RECEIVED FOR REVIEW

MACLEAN, John, M.A., B.Sc.: *Descriptive Mathematics*, Macmillan and Co., 1935; 8°, pp. xvi + 143. Price Rs. 2-8-0.

MILLER, George Abraham, *Collected Works*, University of Illinois Press, 4°, pp. xi + 475.

ALTSHILLER-COURT, Nathan: *Modern Pure Solid Geometry*, Macmillan & Co., New York, 1935, pp. x + 311. Price $ 3.90.

WALSH, J. L. *Interpolation and Approximation by rational functions in the complex domain*, American Mathematical Society, 1935, pp. ix + 382. Price $ 5.00.

ANNOUNCEMENTS AND NEWS

Members of the Society, and subscribers are reminded that all subscriptions are payable in advance at the beginning of the calendar year. Remittances should be sent to the Treasurer, L. N. Subramanyam, M.A., Madras Christian College, Madras.

We are glad to announce that Prof. S. M. Shah of Aligarh has become a life member of the Society.

An International Mathematical Congress, under the presidency of Professors Alf Guldburg and Carl Stormer, will be held at Oslo, Norway, between July 13th and 18th 1936. The congress is organised under seven different sections. (1) Algebra and Theory of Numbers. (2) Analysis. (3) Geometry and Topology. (4) Theory of Probability, Insurance, Statistics. (5) Astronomy. (6) Mechanics and Mathematical Physics. (7) Philosophy, History and Education. Amongst the prominent mathematicians invited to deliver addresses are Professors Birkhoff, Oswald Veblen, Oystein Ore and Norbert Wiener.

At the June commencement of the University of Wisconsin, the degree of Doctor of Science was conferred upon Prof. G. A. Bliss, Head of the department of Mathematics of the University of Chicago.

Prof. Ejnar Hertzsprang has been appointed Director of Leyden Observatory as successor to William de Sitter.

Mr. Pendlebury who completes the 50th year of the Honorary Secretaryship of the London Mathematical Association in January next will be presented at its annual meeting with a piece of plate with a suitable inscription.

The following special lectures will be delivered under the auspices of the Lucknow University.

(1) Jan. 4 and 5, 1936. 'Numbers' by Mr. R. D. Misra, Lecturer in Mathematics. (2) Jan. 19, 20 and 21. "Various Theories of Integration" by Dr. Lakshmi Narain, Reader in Mathematics.

Emmy Noether, daughter of Max Noether, and perhaps the greatest mathematician of her sex, well known for her axiomatic treatment of ideals and of non-commutative Algebra died on April 14, 1935 at the age of 54. An obituary notice and an evaluation of her life work will be found in Weyl's memorial address delivered in Bryn Mawr College, and printed in the *Scripta Mathematica* of July 1935.

Dr. Tsuruichi Hayashi, Editor of the Tohoku Mathematical Journal, suddenly died of cardiorrhexis, on the 4th October at the age of 63 while he was inspecting the Matsue High School as a Government Inspector of the Educational department.

We regret to announce the death on 22-1-1936 at Chitoor, near Madras, of V. Ramaswamy Aiyar, Founder of the Indian Mathematical Society. A detailed obituary notice will appear in the next issue.

OBITUARY

P. V. SESHU AIYAR

Jan. 1872—Oct. 1935

Peruvemba Venkateswara Aiyar Seshu Aiyar was born in a poor brahmin family and felt early in life the pinch of poverty which is the lot of so many Indian boys. His father was a petty farmer and his mother was blind from an attack of small-pox, so that young "Seshu" had to help his mother in the kitchen before he could trudge to the nearest school more than three miles away. He could only return home in the evening after the day's studies were over. These privations must have undermined his health for at the age of nine we find him laid up with a violent attack of typhoid. His health which was never quite robust, gave him considerable anxiety in later life owing to recurring attacks of Neuralgia, and could be maintained only by careful dieting and abstemonious habits. He finished his secondary education in Palghat and in Calicut where he matriculated in 1887 and completed the First in Arts course in 1889 as a stipendiary student. In 1891 he took the Bachelor's Degree from the Madras Christian College where he attracted the attention of the veteran educationist Dr. William Miller, but his studies for the M.A. Degree had to be given up in 1893 owing to a breakdown in health.

Seshu Aiyar entered the field of Education as headmaster of his own Native High School at Palghat, but was soon called to State service as lecturer in the Government College at Kumbakonam—a place which sets high value on mathematical studies and proudly styles itself the Cambridge of South India. It was to this college that Rao Bahadur Seshu Aiyar, the veteran educationist, returned as Principal on the eve of his retirement and

it was also here that he met Ramanujan first as a student and failed F.A. who begged for such physical conditions as would make work in Mathematics possible. One recalls with gratitude and j despite the strangeness of the request, Ramanujan met with what he needed most, viz. understanding, sympathy and an appreciation c potentialities. The Cambridge University paid Seshu Aiyar a well-ı tribute in associating his name with Ramanujan's collected papers.

It was, however, in the Presidency College, Madras, where Seshu Aiy lectured for nearly 15 years, as Lecturer and later as Professor of Applic Mathematics that he felt most in his element. Batches of brilliant Honours Students attended his lectures and few indeed could fail to catch something of his mathematical fervour. His lectures were valued for their clarity and methodical exposition. He set great store by rigour and was profoundly influenced by the French masters Jordan and Picard. To those of us not familiar with such a treatment. it was something of a revelation to see a proof leisurely unveil itself, and take shape like a magic structure, here a lemma and there another, while slowly the invisible bonds reveal themselves and weld them all into a harmonious whole.

Seshu Aiyar was born too early to have come under the influence of research ideals except late in life ; but his strong sense of duty and his high intellectual abilities urged him in spite of poor health, to keep himself well informed of leading developments, particularly in the foundations of Analysis which was his favourite field. With this end in view he acquired, a working knowledge of French and German to study books at firsthand and hence his passion for rigour. He was impatient of a loose step or of a plausible proof at a time when both were not uncommon in teaching and in text-books and the unhappy delinquent felt something of his *furor mathematicus*, especially if he came to seek some favours. This incessant appeal for accuracy and rigour carried on with a missionary zeal did much to stir the pedagogic conscience of South India, especially as Seshu Aiyar himself wrote a text-book on Elementary Mathematics to illustrate his principles.

To the Indian Mathematical Society of which he was a foundation member and at various times Committee Member, Secretary, and President, his services would be difficult to evaluate. It was his courage and faith that made possible the Conference of the Society at Madras in 1919, the parent of all the biennial conferences which have followed thereafter. He occupied positions of influence and was known to have strong views on the duty of teachers towards the Society and this may not be unconnected with a strong demand for membership of the Society from aspirants for university examinerships !

Seshu Aiyar's interests and activities were many-sided. He took a leading part in the creation of a Research department in the Madras University and was interested in educational problems to the very last. He was deeply interested in religious studies but not in a blind or sectarian spirit and, at the Bombay conference, he delivered a lecture on Mathematics and Religion where

out how Cantor's Theory of Transfinite numbers helped in under- the transcendental element in Upanishadic literature. He was a social reformer and was an active supporter of the Sri Ramakrishna or Harijan children. The Peruvemba village Association of which a President and even the village school benefited by his guidance.

nd now that kindly and generous soul has completed its cycle of activities, according to the beliefs which he held in common with other Hindus, has ken, at the bidding of the great Artificer, more auspicious forms appropriate o its further evolution.

तद्यथा पेशस्कारी पेशसो मात्रामुपादायान्यन्नवतरं कल्याणतरं रूपं तनुत एवमेवाय-मात्मेदं शरीरं निहत्याविद्यां गमयित्वान्यन्नवतरं कल्याणतरं रूपं कुरुते ।

—*Brihadaranyakopanishad.*

A. NARASINGA RAO.

* * *

KAVASJI JAMSEDJI SANJANA

We very much regret to record the death at the age of seventy-five of Principal Kavasji Jamsedji Sanjana at Bhavnagar, Kathiawar. He was one of the foundation members of the Indian Mathematical Society and to the end of his days preserved his interest in the subject and contributed numerous problems and notes to our journal.

Sanjana was a Parsee from Poona and graduated from the Deccan College. He had a most distinguished academical career crowned by the Chancellor's Medal in Mathematics at the M.A. in 1878. It is worthy of note that unlike other Parsi students he studied Sanskrit for his second language and was almost as competent in that subject as in Mathematics; one of his colleagues in the class being the late Vaman Shivram Apte, the first principal of the Fergusson College, Poona, and the author of several well-known dictionaries of Sanskrit and the Guide to Sanskrit composition which, I believe, is still largely used by students of Sanskrit. Sanjana, I am told, occasionally took Sanskrit classes also in the Samaldas College. He was also perfectly conversant with Marathi in addition to the usual Parsi language Gujarati. After serving for some years in the Engineering College, Poona, he joined Samaldas College, Bhavnagar, first as professor of Mathematics and in 1914 became its Principal. Under him that college, though located in a comparatively out-of-the-way place, greatly prospered and his name is deeply venerated by all Kathiawar. In his later days and especially after his retirement in 1922 he was handicapped by failing eyesight ending in total blindness in his last years. He still retained his cheerful and buoyant spirits and with the help of an assistant continued to work problems and propose new ones for solution in the Society's Journal.

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